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Variational formulation and gauge symmetries of an ideal fluid

— *Lagrangian description and Eulerian description* —

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Abstract

On the basis of gauge principle in the field theory, a new variational formulation is presented for flows of an ideal fluid. The fluid is defined thermodynamically by a mass density and an entropy density invariant along particle trajectories (by definition of ideal fluid). Flow fields are characterized by symmetries of translation and rotation. A Lagrangian functional is defined in terms of kinetic energy and internal energy, and the action is defined by its integral with respect to time. Noether's theorem leads to Euler's equation of motion and an energy equation. Requirement of the Lagrangian with respect to rotational gauge transformations of particle coordinates (*i.e.* in Lagrange space) results in the invariance of vorticity transformed to the Lagrange space. This implies that the vorticity is a gauge field.

Taking into account invariances of mass, entropy and vorticity in the Lagrange space, one can introduce three additional Lagrangians of the form of total time derivative. The variational principle of the extended action leads to the the continuity equation, the entropy conservation equation, and the vorticity equation. Rotational component of velocity is defined naturally with this formulation. In addition, there exits a close relation between the helicity and the Lagrangian associated with the vorticity, both of which are regarded as Chern-Simons invariants.

Present formulation provides a basis on which the transformation between the Lagrangian space and the Eulerian space is determined uniquely. In most of traditional formualtions, the continuity equation and the entropy equaiton are taken into account as constraints by using Lagrange multiplier whose physical meaning is not clear. Thus, present formulation is consistent as a whole for description of flows of an ideal fluid.

Key words: Gauge priciple, Variational formulation, Ideal fluid, Vorticity, Chern-Simons term

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1 Introduction

Fluid mechanics is a field theory of Newtonian mechanics with Galilean symmetry. It should be covariant under transformations of the Galilei group. Two symmetries (*i.e.* transformation invariances) are known as subgroups of the Galilei group: translation (space and time) and space-rotation. A guiding principle in physics is that physical laws should be expressed in a form that is independent of any particular coordinate system. In the present paper, we seek to formulate flows of an ideal fluid with the action principle which has a formal equivalence with the gauge theory in physics (*e.g.* Weinberg 1995). The gauge theory provides a basis for reflection on similarity between fluid mechanics and other physical fields.

Studies following the above theme have been carried out recently (Kambe 2003; Kambe 2007; Kambe 2008a, b), in order to investigate possiblity whether flow fields of a fluid can be formualted according to the gauge theory. Outcome of those preliminary attempts is satisfactory and encourages to proceed to more fundamental formulation from the view point of gauge symmetries of flow fields. Present paper is an endeavor according to the above philosophy.

First, a Lagrangian functional is defined (as in Seliger & Witham 1968) by a combination of total kinetic energy and internal energy in the space of particle coordinates (denoted as $\boldsymbol{a}$ space), called also as *Lagrangian description* (in §2.2). The action is given by its time integral. Noether's theorem leads to the equation of motion and an energy equation (§2.3). In most of traditional formulations (Herivel 1955; Serrin 1959; Eckart 1960; Seliger & Witham 1968; Salmon 1988), the continuity equation and the equation of isentropy [1] are added as constraints by using Lagrange multipliers to the action integral. However, in the new formulation, those equations are derived from the variational principle, rather than being given as constraints. To that end, additional Lagrangians are introduced by symmetry consideration.

In the present formulation, the newly added Lagrangians are determined so that it is invariant with respect to both translational and rotational transformations. For that, the following three properties are taken into consideration. (a) Fluid mass is an invariant of motion. (b) Entropy per unit mass is another invariant of motion by the definition of an ideal fluid. And, (c) vorticity $\boldsymbol{\Omega}_a$ in the $\boldsymbol{a}$-space is invariant, which is derived by requirement of gauge invariance of the action in the $\boldsymbol{a}$-space (§3.2). Thus, three Lagrangians are newly introduced (§2.5 and §3.4), all of which have characteristic forms such that they are of the form of total time derivative. Hence, their action integrals are integrated

[1] *Homentropy* means that fluid entropy is uniform throughout the space, while *isentropy* means that each fluid particle keeps its entropy value along its trajectory, but fluid is not necessarily homentropic.

with respect to time in the Lagrangian description. So that, the newly added terms do not influence the Euler-Lagrange equation in the $\boldsymbol{a}$-space. However, in the space of Eulerian coordinates denoted by $\boldsymbol{x}$, the continuity equation and an entropy equation are derived from the action principle (§5.1), since the time derivative of the Lagrangian description is replaced with the material derivative including convection velocity and mass density is a function of $\boldsymbol{x}$ and time t. Noether's theorem in the $\boldsymbol{x}$-space reduces to the conservation equation of momentum (§5.2).

According to variations of the Eulerian description, if the fluid is homentropic, the action principle of an ideal fluid results in potential flows (§5.1; Herivel 1955). It is generally understood that, even in such a homentropic fluid, it should be possible to have rotational flows. In fact, this is a long-standing problem. Lin (1963) tried to resolve this difficulty by introducing *Lin's constraint* as a side condition, imposing invariance of the Lagrange coordinates $\boldsymbol{a}$ along particle trajectories, and the constraints are taken into account by using Lagrange multipliers, called potentials. However, physical significance of those potentials is not clear (Bretherton 1970): for example, it is not clear why the Lagrange multiplier for the continuity equation becomes the velocity potential for flows of a homentropic fluid.

The present formulation provides a key to resolve the above issue. Namely, a rotational component of velocity field is derived in §5.1 naturally from the Lagrangian L_A associated with the third property (c). In the case of an incompressible fluid, the rotational component $\boldsymbol{w}_*$ is represented as $\boldsymbol{w}_* = \nabla \times \boldsymbol{\Psi}_*$ (§6.3), as is usually assumed in incompressible fluid mechanics. The property (c) implies that the vorticity is a gauge field associated with the rotation symmetry of the flow field (§3.2), and the vorticity equation is derived from L_A in §5.1.

A gauge theory of rotation invariant Lagrangian with an internal $O(3)$-symmetry was studied for the Bohr model of nuclear collective rotation of a finite number of modes (Fujikawa & Ui 1986). There is a similarity between this system (of five field variables) and the fluid flows (of infinite dimensions). In particular, both systems are considered a dynamical system. The gauge field of the nuclear collective rotation was found to be the angular velocity.

Helicity $\mathcal{H}$ is defined by a space integral of inner product of vorticity and velocity, and known to be a topological invariant. This is defined mathematically as a secondary invariant satisfying two conditions (Chern 1979, Appendix 4), called also the *Chern-Simons invariant* (Jackiw 1984). It is shown in §6.1 that $\mathcal{H}$ in fact satisfies coresponding two equations, (78) and (80). The helicity $\mathcal{H}$ and the Lagrangian L_A are closely related to each other in the sense that, when one exists, then the other also exists (§6.4). It is verified that both of the integrals are time-invariant, by using the time-invariance of $\boldsymbol{\Omega}_a$. This implies that the Lagrangian L_A is also the secondary invariant (*i.e.* another one of Chern-Simons invariants). In fluid mechanics, although the helicity $\mathcal{H}$ is investigated in detail, the Lagrangian L_A has never been considered explicitly.

Local transformation from the Lagrangian $\boldsymbol{a}$-space to the Eulerian $\boldsymbol{x}$-space is determined by nine elements of the 3×3 matrix $\partial x^k / \partial a^l$ at each point. Three relations between the vorticity components in the $\boldsymbol{a}$-space and those in the $\boldsymbol{x}$-space are required for unique transformation, in addition to the well-known six relations of velocity and acceleration in both spaces. This fact that is not awared sufficiently so far is considered in §2.4 and §7.

As a note, a recent study (Constatin 2001) of an Eulerian-Lagrangian local approach for incompressible fluids is to be remarked, although this is neither based on variational formulation, nor transformation uniquness is considered. Rather, this paper is interested in an open question if any solution of blow-up to the Euler equation. Another work to be noted is the monograph (Jackiw 2002), applying the ideas of particle physics to fluid mechanics in terms of Hamiltonian and Poisson brackets both relativistically and nonrelativistically. Relativistic Lagrangian approach is also taken by Soper (1976).

Last section 8 investigates whether the Lagrangian density Λ defined in the present analysis has in fact symmetries of translation and rotation. It is confirmed there that the gauge symmetries are satisfied by Λ. Thus, the present formulation is reasonable in physical senses and consistent as a whole for description of flow fields of an ideal fluid.

2 Equations of Lagrange description

2.1 Particle coordinates and definition of an ideal fluid

First of all, we consider a variational problem of the action defined by using a Lagrangian functional represented with the particle coordinates, also called as Lagrangian coordinates, which are denoted as $\boldsymbol{a} = (a^1, a^2, a^3) = (a, b, c)$. Time variable τ is used in combination with (a, b, c).[2]

Physical space coordinates (*i.e.* Eulerian coordinates) are written as $\boldsymbol{x} = (x, y, z) = (x^1, x^2, x^3)$ with t denoting the time. Position coordinates of a fluid particle of the label $\boldsymbol{a}$ are denoted by $X^k(a^\mu)$, or $\boldsymbol{X}(\tau, \boldsymbol{a}) = (X^k) =$

[2] When independent variables are written as $a^\mu = (\tau, a^1, a^2, a^3)$, the parameter μ (or a greek letter) takes the values of 0,1,2,3, where $a^0 = \tau$. Roman letters such as k or l take 1,2,3.

$(X^1, X^2, X^3) = (X, Y, Z)$. Particle velocity is given by $v^k = \partial_\tau X^k = \partial X^k / \partial a^0$, or X^k_0.

Mass elelment dm in a volume elelemt of physical space $\mathrm{d}^3\boldsymbol{x} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is defined by $\mathrm{d}^3\boldsymbol{a} = \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c$ in the space of Lagrange coordinates. Namely,

$$dm = \mathrm{d}^3\boldsymbol{a} = \rho\,\mathrm{d}^3\boldsymbol{x}, \qquad \rho: \text{ mass density.}$$

The mass dm is an invariant of motion. Hence the following must be always satisfied (where $\partial_\tau = \partial/\partial\tau$) :

$$\partial_\tau(\mathrm{d}m) \equiv \partial_\tau(\mathrm{d}^3\boldsymbol{a}) = 0\,. \tag{1}$$

From the relation $dm = \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c = \rho\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z$, the mass density is given by

$$\rho = \frac{1}{J}, \quad J = \frac{\partial(X^k)}{\partial(a^l)} = \frac{\partial(X^1, X^2, X^3)}{\partial(a^1, a^2, a^3)} = \frac{\partial(X, Y, Z)}{\partial(a, b, c)}\,, \tag{2}$$

where J is the *Jacobian* of the transformation, [$\boldsymbol{a}$ *vs.* $\boldsymbol{x}$].

An ideal fluid is defined by one in which there is no dissipaton of kinetic energy during motion. Therefore, the entropy s per unit mass is invariant, *i.e.* τ-independent and $s = s(a, b, c)$. Thus, we have the following:

$$\partial_\tau\, s = 0\,. \tag{3}$$

According to the thermodynamics with the internal energy ϵ per unit mass, the enthalpy h is defined by $h = \epsilon + p/\rho$ per unit mass. For a density change $\delta\rho$ and an entropy change δs, changes of ϵ and h are given by

$$\delta\epsilon = (p/\rho^2)\delta\rho + T\delta s, \qquad \delta h = (1/\rho)\delta p + T\delta s,$$

where p is the pressure, and T the temperature. Setting $\delta s = 0$, we have

$$\delta\epsilon = (\delta\epsilon)_s = \frac{p}{\rho^2}\,\delta\rho, \qquad \delta h = (\delta h)_s = \frac{1}{\rho}\delta p, \tag{4}$$

where $(\,\cdot\,)_s$ denotes variation with s fixed. These are relations between variations of state variables of an ideal fluid.

2.2 Lagrangian

The Lagrangian expressed in terms of the particle coordinates is defined by[3]

$$L_{\mathrm{T}} = \int_{M_a} \tfrac{1}{2}\, X_\tau^k\, X_\tau^k\, \mathrm{d}^3\boldsymbol{a} - \int_{M_a} \epsilon(\rho, s)\, \mathrm{d}^3\boldsymbol{a}\,. \tag{5}$$

Action integral I is given by

$$I = \int_{M_a \oplus I_\tau} \Lambda(X_\mu^k, X^k)\; \mathrm{d}^4 a, \qquad \mathrm{d}^4 a = \mathrm{d}\tau\, \mathrm{d}^3\boldsymbol{a}, \tag{6}$$

where the Lagrangian density Λ is defined by

$$\Lambda = \Lambda(X_\mu^k, X^k) \equiv \tfrac{1}{2}\, X_0^k\, X_0^k - \epsilon(X_l^k, X^k) \tag{7}$$

$(k, l = 1, 2, 3)$. Since the density ρ depends on $X_l^k \equiv \partial X^k / \partial a^l$ from (2) and s depends on $X^k(\boldsymbol{a})$, we have $\epsilon(\rho, s) = \epsilon(X_l^k, X^k)$ in the last equation. Integration domain of $\boldsymbol{a}$ is denoted by M_a chosen arbitrarily, and the interval of τ-integration is $I_\tau = [\tau_1,\, \tau_2]$.

2.3 Euler-Lagrange equations

By the action principle, it is required that the action I is invariant for the variation of Lagrangian density $\Lambda(X_\mu^k, X^k)$ with respect to arbitrary transformation $X^k \to X^k + \xi^k$, where $\xi^k(a^\mu)$ $(k = 1, 2, 3)$ are arbitrary functions of a^μ. This results in the following Euler-Lagrange equation (Kambe 2008a):

$$\frac{\partial}{\partial a^\mu}\left(\frac{\partial \Lambda}{\partial X_\mu^k}\right) - \frac{\partial \Lambda}{\partial X^k} = 0, \qquad\qquad \mu = 0, \cdots, 3;\;\; k = 1, 2, 3, \tag{8}$$

where $X_\mu^k = \partial X^k / \partial a^\mu$, and the variations $\xi^k(a^\mu)$ are assumed to vanish on the boundary of the domain $M_a \oplus I_\tau$.

In this formulation, an energy-momentum tensor T_μ^ν is defined by

$$T_\mu^\nu \equiv X_\mu^k \left(\frac{\partial \Lambda}{\partial X_\nu^k}\right) - \Lambda\, \delta_\mu^\nu\,, \tag{9}$$

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[3] Traditionally (Herivel 1955; Serrin 1959; Eckart 1960; Seliger & Witham 1968; Salmon 1988), Lagrangian includes constraints with Lagrange multipliers, in addition to L_{T}. In this regard, the present L_{T} is a new form, which will be clarified later in §4.3.

Using this, the Noether theorem is given by Kambe (2008a)

$$\frac{\partial}{\partial a^\nu} T^\nu_\mu = 0 \,. \tag{10}$$

The following ($\boldsymbol{a}$) and ($\boldsymbol{b}$) give components of this equation (Eckart 1960).

($\boldsymbol{a}$) For $\mu \neq 0$ ($a^\mu = \alpha$), the equation $\partial_\nu T^\nu_\mu = 0$ results in the following:

$$\partial_\tau V_\alpha + \partial_\alpha F = 0 \tag{11}$$

$$V_\alpha \equiv X_\alpha X_\tau + Y_\alpha Y_\tau + Z_\alpha Z_\tau, \tag{12}$$

where $F = -\frac{1}{2} v^2 + h$. Remaining (two) equations are obtained by cyclic permutation of α among (a, b, c). Integrating (11) with respect to τ from 0 to t, we obtain the following Weber's equation (Lamb 1932, §15):

$$V_\alpha(t, \boldsymbol{a}) = V_\alpha(0, \boldsymbol{a}) - \partial_\alpha \chi, \tag{13}$$

$$\frac{\partial \chi}{\partial \tau} = -\tfrac{1}{2} v^2 + h \equiv F, \qquad v^2 = (X_\tau)^2 + (Y_\tau)^2 + (Z_\tau)^2. \tag{14}$$

The equation (12) expresses the velocity V_α transformed to the $\boldsymbol{a}$-space (§3.3). Its time evolution is given by (13) and (14) for initial values $V_\alpha(0, \boldsymbol{a})$ and $h(0, \boldsymbol{a})$.

($\boldsymbol{b}$) For $\mu = 0$, we obtain an enrgy equation:

$$\partial_\tau H + \partial_a \left[p \frac{\partial(X, Y, Z)}{\partial(\tau, b, c)} \right] + \partial_b \left[p \frac{\partial(X, Y, Z)}{\partial(a, \tau, c)} \right] + \partial_c \left[p \frac{\partial(X, Y, Z)}{\partial(a, b, \tau)} \right] = 0, \tag{15}$$

where $H = \frac{1}{2} v^2 + \epsilon$. Corresponding equation in the Eulerican description is given by (77) of §5.2.

From (11), we obtain the following equation for the acceleration $\mathcal{A}_\alpha(\tau, \boldsymbol{a})$:

$$\mathcal{A}_\alpha \equiv X_\alpha X_{\tau\tau} + Y_\alpha Y_{\tau\tau} + Z_\alpha Z_{\tau\tau} = -\partial_\alpha h. \tag{16}$$

This is known as the equation of motion in the Lagrange form (Lamb 1932, §13), where the following equation is used for its derivation:

$$\partial_\tau V_\alpha = \mathcal{A}_\alpha + X^k_{\tau\alpha} X^k_\tau = \mathcal{A}_\alpha + \tfrac{1}{2} \partial_\alpha (X^k_\tau)^2 = \mathcal{A}_\alpha + \partial_\alpha (\tfrac{1}{2} v^2). \tag{17}$$

The equation (16) is immediately transformed to

$$X_{\tau\tau} = -\frac{1}{\rho} \partial_x p, \qquad \partial_x p = \frac{\partial \alpha}{\partial x} \frac{\partial p}{\partial \alpha}. \tag{18}$$

Since the term $X_{\tau\tau}$ on the left side is the acceleration of a particle $\boldsymbol{a}$, this equation is nothing but the Euler's equation of motion.

2.4 Freedom of transformation between $\boldsymbol{a}$ and $\boldsymbol{x}$

There is an arbitrariness in the transformation from $\boldsymbol{a}$-space to $\boldsymbol{x}$-space. In regard to the equation (16), the expression of the middle side has a form of inner product of two vectors: a vector of particle acceleration $(X_{\tau\tau}, Y_{\tau\tau}, Z_{\tau\tau})$ in the $\boldsymbol{x}$-space and a direction vector $(X_\alpha, Y_\alpha, Z_\alpha)$ in the $\boldsymbol{x}$-space representing the α-axis of the $\boldsymbol{a}$-space. In other words, the equation (16) is invariant with respect to rotational transformations of the displacement vector $\Delta\boldsymbol{X} = (\Delta X, \Delta Y, \Delta Z)$ of a particle in the $\boldsymbol{x}$-space.

There is the same sort of freedom for the particle velocity $V_\alpha(\tau, \boldsymbol{a})$ of (12) as well. Later, we will consider another (third) transformation of vorticity for the uniqueness of transformation between $\boldsymbol{a}$ and $\boldsymbol{x}$ spaces.

2.5 Trivial Lagrangians

Mass $\mathrm{d}^3\boldsymbol{a}(\boldsymbol{a})$ and entropy $s = s(\boldsymbol{a})$ satisfy the invariance equations (1) and (3). Using these properties, one can introduce the following two Lagrangians:

$$L_\phi = -\int_M \partial_\tau \phi \, \mathrm{d}^3\boldsymbol{a}, \qquad L_\psi = -\int_M s \, \partial_\tau \psi \, \mathrm{d}^3\boldsymbol{a}, \tag{19}$$

where $\phi(\boldsymbol{a}, \tau)$ and $\psi(\boldsymbol{a}, \tau)$ are scalar fields associated with mass and entropy. The Euler-Lagrange equation is unchanged, even if these are added to L_T of (5).

In fact, adding L_ϕ and L_ψ to L_T, we have the following total Lagrangian:

$$L_\mathrm{T}^* = L_\mathrm{T} - \int \partial_\tau \phi \; \mathrm{d}^3\boldsymbol{a} - \int s \, \partial_\tau \psi \; \mathrm{d}^3\boldsymbol{a}. \tag{20}$$

In addition, the action integral is defined by

$$I = \int_{\tau_1}^{\tau_2} L_\mathrm{T}^* \mathrm{d}\tau = \int \mathrm{d}\tau \, L_\mathrm{T} - \int \mathrm{d}\tau \int \partial_\tau \phi \, \mathrm{d}^3\boldsymbol{a} - \int \mathrm{d}\tau \int s \, \partial_\tau \psi \, \mathrm{d}^3\boldsymbol{a}. \tag{21}$$

The second integral on the right hand side $I_\phi = \int \mathrm{d}\tau \int \partial_\tau \phi \, \mathrm{d}^3\boldsymbol{a}$ can be integrated with respect to τ, and may be expressed as $\int [\phi] \mathrm{d}^3\boldsymbol{a}$, where $[\phi] = \phi|_{\tau_2} - \phi|_{\tau_1}$ is the difference of ϕ at two times τ_2 and τ_1 and independent of the times between those two. Likewise, the third integral can be written as $I_\psi = \int [\psi] s \, \mathrm{d}^3\boldsymbol{a}$, since s is independent of τ. This means that the two scalar fields ϕ and ψ do not appear in the Euler-Lagrange equation, which is derived by the variational principle of the action I with varied fields at inner times $\tau \in (\tau_1, \tau_2)$. Namely, the Euler-Lagrange equation is invariant by the addition of I_ϕ and I_ψ. In this regard, L_ϕ and L_ψ may be called as *trivial* Lagrangians. However, these are non-trivial for the variation of fields in the Eulerian space, as will be seen in §5.1.

3 Gauge invariance and differential forms

3.1 Particle permutation

The Lagrangian (5) has a parameter invariance. Consider a transformation of parameters from $\boldsymbol{a} = (a,\, b,\, c)$ to $\boldsymbol{a}' = (a',\, b',\, c') = \boldsymbol{a}'(a, b, c)$. In this transformation, the Jacobian J_a must satify the following:

$$J_a \equiv \frac{\partial(a', b', c')}{\partial(a, b, c)} = 1. \tag{22}$$

For, the mass $\mathrm{d}m$ of any small element $\mathrm{d}^3\boldsymbol{a}$ must be invariant in this transformation. Then, the condition $J_a = 1$ is required, since we must have $\mathrm{d}m = \mathrm{d}^3\boldsymbol{a}$ and $\mathrm{d}m = \mathrm{d}^3\boldsymbol{a}' = J_a \mathrm{d}^3\boldsymbol{a}$.

The parameters (a, b, c) are coordinates of a fluid particle. Hence, the above parameter invariance is understood as the invariance of particle permutation. From another point of view, there exists an arbitrariness for the definition of coordinates. This is understood as a gauge transformation with respect to the coordinate transformation $\boldsymbol{a} \to \boldsymbol{a}'$. Therefore, the above is interpreted as a *gauge invariance* of the Lagrangian (5).

3.2 Gauge invariance in the $\boldsymbol{a}$ space

Provided that the parameter transformation is expressed by $(a^k)' = a^k + \delta a^k$ for small variations δa^k, first-order variation of the Jacobian J_a must satisfy

$$\delta \left(\frac{\partial(a', b', c')}{\partial(a, b, c)} \right) = \frac{\partial}{\partial a^k}\, \delta a^k = 0, \tag{23}$$

from (22). Hence, the varaition vector $\delta\boldsymbol{a} = (\delta a^k)$ can be represented by using a certain vector potential $\delta\boldsymbol{\Phi}$ as

$$\delta\boldsymbol{a} = \nabla_a \times \delta\boldsymbol{\Phi}, \tag{24}$$

where

$$\nabla_a \equiv (\partial_a, \partial_b, \partial_c) = (\partial_{a^1}, \partial_{a^2}, \partial_{a^3}).$$

Because $\mathrm{div}(\mathrm{curl}\,\delta\boldsymbol{\Phi}) = 0$, the equation (23) is satisfied. Since $\mathrm{curl}\,\delta\boldsymbol{a} \neq 0$, the transformation $\boldsymbol{a} \to \boldsymbol{a}'$ includes rotational transformations.

Variation of the action I of (6) can be written as

$$\delta I = \int \delta\Lambda(X^k_\mu, a^k)\, \mathrm{d}^4 a = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \int \Big(X^k_\tau \frac{\partial}{\partial a^l} X^k_\tau - \frac{\partial}{\partial a^l}\,\epsilon \Big)\, \delta a^l\ \mathrm{d}^3\boldsymbol{a}, \tag{25}$$

where, using (12) and (16),

$$X^k_\tau \frac{\partial}{\partial a^l} X^k_\tau = X^k_\tau \frac{\partial}{\partial \tau} X^k_l = -X^k_{\tau\tau} X^k_l + \frac{\partial}{\partial \tau}\Big(X^k_\tau X^k_l \Big) = \partial_l\, h + \partial_\tau V_l.$$

Substituting this into (25), we obtain

$$\delta I = \int \mathrm{d}\tau \int \Big(\partial_\tau V_l + \frac{\partial}{\partial a^l}\,(h - \epsilon) \Big)\, \delta a^l\ \mathrm{d}^3\boldsymbol{a}.$$

Substituting (24),

$$\delta I = \int \mathrm{d}\tau \int_{M_a} \Big(\partial_\tau \boldsymbol{V}_a + \nabla_a (h - \epsilon) \Big) \cdot (\nabla_a \times \delta\boldsymbol{\Phi})\ \mathrm{d}^3\boldsymbol{a},$$

where $\boldsymbol{V}_a = (V_a, V_b, V_c)$. Carrying out integration by parts with respect to the coordinate $\boldsymbol{a}$, we obtain

$$\delta I = \int \mathrm{d}\tau \int_{M_a} \Big(\partial_\tau (\nabla_a \times \boldsymbol{V}_a) \Big) \cdot \delta\boldsymbol{\Phi}\ \mathrm{d}^3\boldsymbol{a} + \int \mathrm{d}\tau\ [\mathrm{Int}_S], \tag{26}$$

where Int_S denotes integrations over the surface S_a bounding M_a. Invriance of the action, $\delta I = 0$ for arbitrary variation $\delta\boldsymbol{\Phi}$, requires the following:

$$\partial_\tau\, \boldsymbol{\Omega}_a = 0, \qquad \text{where} \quad \boldsymbol{\Omega}_a \equiv \nabla_a \times \boldsymbol{V}_a. \tag{27}$$

It will be shown in the next section that $\boldsymbol{\Omega}_a$ is the vorticity transformed to the $\boldsymbol{a}$-space.

From the above consideration, invariance of I means the invariance with respect to particle permutation, from which time-invariance of the vorticity in the $\boldsymbol{a}$-space is derived by the requirement of gauge invariance of the action. This implies that the vorticity $\boldsymbol{\Omega}_a$ in the $\boldsymbol{a}$-space is in fact a *gauge field*.

We consider differential forms in terms of both Lagrangian coordintaes (a,b,c) and Eulerian coordintaes (x,y,z). The variables V_α $(\alpha = a,b,c)$ defined by (12) are the velocity components transformed to the $\boldsymbol{a}$-space. The velocities in the $\boldsymbol{x}$-space are written as $\boldsymbol{v} = (u,v,w) = (X_\tau, Y_\tau, Z_\tau)$. Then, one can define one-form $\mathcal{V}^1$ with [4]

$$\mathcal{V}^1 = u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z \qquad (\text{written as } \ \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x}) \tag{28}$$
$$= V_a\,\mathrm{d}a + V_b\,\mathrm{d}b + V_c\,\mathrm{d}c \qquad (\text{written as } \ \boldsymbol{V}_a\cdot\mathrm{d}\boldsymbol{a}). \tag{29}$$

Owing to the equation (12), it is seen that the expression (28) is equal to (29). Applying exterior differentiation to (28) and (29), we obtain two-form $\Omega^2 = \mathrm{d}\mathcal{V}^1$, defined by

$$\Omega^2 = \mathrm{d}\mathcal{V}^1 = \Omega_a\,\mathrm{d}b\wedge\mathrm{d}c + \Omega_b\,\mathrm{d}c\wedge\mathrm{d}a + \Omega_c\,\mathrm{d}a\wedge\mathrm{d}b = \boldsymbol{\Omega}_a\cdot\boldsymbol{S}^2 \tag{30}$$
$$= \omega_x\,\mathrm{d}y\wedge\mathrm{d}z + \omega_y\,\mathrm{d}z\wedge\mathrm{d}x + \omega_z\,\mathrm{d}x\wedge\mathrm{d}y = \boldsymbol{\omega}\cdot\boldsymbol{s}^2, \tag{31}$$
$$\boldsymbol{\Omega}_a \equiv \nabla_a \times \boldsymbol{V}_a = (\Omega_a, \Omega_b, \Omega_c),$$
$$\boldsymbol{\omega} \equiv \nabla\times\boldsymbol{v} = (\omega_x, \omega_y, \omega_z) \qquad (\text{vorticity}),$$

where $\boldsymbol{s}^2$ and $\boldsymbol{S}^2$ are defined at the footnote. It is seen that $\boldsymbol{\Omega}_a$ is the vorticity transformed to the $\boldsymbol{a}$-space.

Writing (11) in the one-form style, we have

$$\partial_\tau \mathcal{V}^1 + \mathrm{d}F = 0. \tag{32}$$

Taking exterior derivative, we obtain the following two-form equation:

$$\partial_\tau\,\mathrm{d}\mathcal{V}^1 + \mathrm{d}^2F = \partial_\tau\Omega^2 = 0, \qquad (\mathrm{d}\mathcal{V}^1 = \Omega^2,\ \mathrm{d}^2F = 0). \tag{33}$$

The equation $\partial_\tau\Omega^2 = 0$ is equivalent to $\partial_\tau\,\boldsymbol{\Omega}_a = 0$ of (27). Namely, we have the expression $\boldsymbol{\Omega}_a = \boldsymbol{\Omega}_a(\boldsymbol{a})$.

3.4 Another trivial Lagrangian

Introducing a vector potential $\boldsymbol{A}_a = (A_a, A_b, A_c)$ in the $\boldsymbol{a}$-space, we define a corresponding one-form as follow:

$$A^1 = A_a\,\mathrm{d}a + A_b\,\mathrm{d}b + A_c\,\mathrm{d}c$$

[4] $\mathrm{d}\boldsymbol{x} = (\mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z)$, $\mathrm{d}\boldsymbol{a} = (\mathrm{d}a,\ \mathrm{d}b,\ \mathrm{d}c)$, and $\boldsymbol{s}^2 = (\mathrm{d}y\wedge\mathrm{d}z,\ \mathrm{d}z\wedge\mathrm{d}x,\ \mathrm{d}x\wedge\mathrm{d}y)$, $\boldsymbol{S}^2 = (\mathrm{d}b\wedge\mathrm{d}c,\ \mathrm{d}c\wedge\mathrm{d}a,\ \mathrm{d}a\wedge\mathrm{d}b)$.

Using this, we define a three-form K^3 by the exterior product of A^1 and Ω^2:

$$K^3 = A^1 \wedge \Omega^2 = \langle \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3\boldsymbol{a}\,, \qquad \mathrm{d}^3\boldsymbol{a} = \mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c, \tag{34}$$

where $\langle \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle = A_a\Omega_a + A_b\Omega_b + A_c\Omega_c$.

Let us regard K^3 as a three-form over the four dimensional manifold $(\tau, \boldsymbol{a})$ and define the external derivative $\mathbf{d}_4$ with

$$\mathbf{d}_4 \equiv \mathrm{d}\tau \wedge \partial_\tau + \mathrm{d}\,, \tag{35}$$

by writing its components, where d denotes the external derivative in the three-space $\boldsymbol{a} = (a, b, c)$. Taking the differentiation $\mathbf{d}_4$ of K^3,

$$R^4 \equiv \mathbf{d}_4 K^3 = \mathbf{d}_4 A^1 \wedge \Omega^2 - A^1 \wedge \mathbf{d}_4 \Omega^2 = \mathrm{d}\tau \wedge (\partial_\tau A^1) \wedge \Omega^2. \tag{36}$$

where we used $\mathbf{d}_4\Omega^2 = 0$ since $\partial_\tau\Omega^2 = 0$ and $\mathrm{d}\Omega^2 = \mathrm{d}^2\mathcal{V}^1 = 0$, and used $\mathrm{d}A^1 \wedge \Omega^2 = 0$ since this is a four-form on the three-space (a, b, c). Like (34), we obtain

$$R^4 = \langle \partial_\tau \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}\tau \wedge \mathrm{d}^3\boldsymbol{a} = \partial_\tau \langle \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle \;\; \mathrm{d}\tau \wedge \mathrm{d}^3\boldsymbol{a}. \tag{37}$$

where (27) is used.

Thus, owing to the invariance of the vorticity $\partial_\tau \boldsymbol{\Omega}_a = 0$, one can define another trivial Lagrangian L_A by an integral of an exact form $-\mathbf{d}_4 K^3$ as follows:

$$L_A = -\int_M \langle \partial_\tau \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3\boldsymbol{a} = -\partial_\tau \int_M \langle \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3\boldsymbol{a}, \tag{38}$$

where $\boldsymbol{\Omega}_a$ is a function of $\boldsymbol{a}$ only. The negative sign is added as a matter of convenience.

This new Lagrangian L_A is invariant with respect to both translational and rotational transformations, as explained in §8 later. Obviously, the new action obtained from L_A can be integrated with respect to τ. Hence, likewise the cases of L_ϕ and L_ψ of (19), the vector potential $\boldsymbol{A}_a$ does not appear in the Euler-Lagrange equation (in §2.3). However, as considered below, the Lagrangian L_A is non-trivial in the variations represented with variables and fields in the $\boldsymbol{x}$-space.

4 Equations of Euler description

4.1 Eulerian description

Independent variables in the Eulerian description are (t, x, y, z), the variables in the physical space. According to the gauge-theoretic formulations of fluid motion (Kambe 2007; Kambe 2008a, b), the time derivative ∂_τ is represented by the following operator D_t:

$$\partial_\tau = \mathrm{D}_t, \qquad \mathrm{D}_t \equiv \partial_t + v^k \partial_k = \partial_t + \boldsymbol{v} \cdot \nabla \,,$$

where the vector $\boldsymbol{v} = (u, v, w)$ denotes the particle velocity $\partial_\tau \boldsymbol{X}$:

$$\boldsymbol{v}(\boldsymbol{X}, t) = \partial_\tau \boldsymbol{X}(\tau, \boldsymbol{a}) = \frac{\mathrm{d}}{\mathrm{d}\tau} \boldsymbol{X}_a(\tau).$$

The velocity is also defined by

$$\boldsymbol{v} = \mathrm{D}_t \boldsymbol{x} \,. \tag{39}$$

These are considered later in §8.2 together with the gauge invariance of D_t.

4.2 L_A in Eulerian space

In §3.3, the velocity one-form $\mathcal{V}^1$ and vorticity two-form Ω^2 are defined. Next, with introducing a vector potential $\boldsymbol{A} = (A_x, A_y, A_z)$ in the $\boldsymbol{x}$-space, let us define its one-form version by

$$A^1 = A_x \, \mathrm{d}x + A_y \, \mathrm{d}y + A_z \, \mathrm{d}z. \tag{40}$$

Taking exterior product of A^1 with Ω^2,

$$A^1 \wedge \Omega^2 = \langle \boldsymbol{A}, \, \boldsymbol{\omega} \rangle \, \mathrm{d}^3 \boldsymbol{x} \,, \tag{41}$$

where $\langle \boldsymbol{A}, \, \boldsymbol{\omega} \rangle = A_x \omega_x + A_y \omega_y + A_z \omega_z$. Equating the two equivalent three-forms (34) and (41), we obtain $\langle \boldsymbol{A}_a, \boldsymbol{\Omega}_a \rangle \mathrm{d}^3 \boldsymbol{a} = \langle \boldsymbol{A}, \boldsymbol{\omega} \rangle \mathrm{d}^3 \boldsymbol{x}$.

Let us define a Lie-derivative operator $\mathcal{L}_W$ by

$$\partial_\tau \equiv \mathcal{L}_W = \mathcal{L}_{\partial_t + V} = \partial_t + \mathcal{L}_V, \tag{42}$$

where $V = u\partial_x + v\partial_y + w\partial_z$. Since $\partial_\tau \Omega^2 = 0$, we obtain the following:

$$0 = \partial_\tau \Omega^2 = \mathcal{L}_W \Omega^2 = \partial_t \Omega^2 + \mathcal{L}_V \Omega^2 = \mathcal{L}_W^*[\boldsymbol{\omega}] \cdot \boldsymbol{s}^2, \tag{43}$$

where, by using $\nabla \cdot \boldsymbol{\omega} = 0$,

$$\mathcal{L}_W^*[\boldsymbol{\omega}] \equiv \partial_t \boldsymbol{\omega} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{v}). \tag{44}$$

Hence, we obtain the following vorticity equation:

$$\partial_t \boldsymbol{\omega} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{v}) = 0. \tag{45}$$

Using (42) and (43), we obtain

$$\partial_\tau \left[\langle \boldsymbol{A}_a, \boldsymbol{\Omega}_a \rangle \, \mathrm{d}^3 \boldsymbol{a} \right] = \mathcal{L}_W \left[A^1 \wedge \Omega^2 \right] = \left[\mathcal{L}_W \, A^1 \right] \wedge \Omega^2. \tag{46}$$

Furthermore, the right hand side is given by $\langle \mathcal{L}_W \boldsymbol{A}, \boldsymbol{\omega} \rangle \, \mathrm{d}^3 \boldsymbol{x}$. Lie derivative of the one-form A^1 is given by

$$\mathcal{L}_W A^1 \equiv (\partial_t A_i + v^k \partial_k A_i + A_k \partial_i v^k) \, \mathrm{d}x^i = -\Psi_i \, \mathrm{d}x^i = -\Psi^1, \tag{47}$$
$$-\Psi_i \equiv (\mathcal{L}_W \boldsymbol{A})_i = \partial_t A_i + v^k \partial_k A_i + A_k \partial_i v^k \tag{48}$$

(see *e.g.* Frankel 1997, §4.2). Thus, the Lagrangian L_A of (38) can be represented in the $\boldsymbol{x}$-space as follows:

$$L_A = -\int_M \langle \mathcal{L}_W \boldsymbol{A}, \boldsymbol{\omega} \rangle \, \mathrm{d}^3 \boldsymbol{x} = \int_M \langle \boldsymbol{\Psi}, \boldsymbol{\omega} \rangle \, \mathrm{d}^3 \boldsymbol{x}, \qquad \boldsymbol{\Psi} \equiv -\mathcal{L}_W \boldsymbol{A}. \tag{49}$$

Substituting $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ and carrying out integration by parts, we obtain

$$L_A = \int_M \langle \nabla \times \boldsymbol{\Psi}, \boldsymbol{v} \rangle \, \mathrm{d}^3 \boldsymbol{x} + \mathrm{Int}_S = -\int_M \rho \, \langle \boldsymbol{b}, \boldsymbol{v} \rangle \, \mathrm{d}^3 \boldsymbol{x} + \mathrm{Int}_S, \tag{50}$$

where Int_S denotes integrations over the surface S bounding M. This is the Lagrangian L_A in the Eulerian space, where

$$\rho \, \boldsymbol{b} \equiv -\nabla \times \boldsymbol{\Psi} = \nabla \times (\mathcal{L}_W \boldsymbol{A}). \tag{51}$$

Substituting (48) into (49), and carrying out integration by parts, we obtain

$$L_A = \int_M \langle \boldsymbol{A}, \mathcal{L}_W^*[\boldsymbol{\omega}] \rangle \, \mathrm{d}^3 \boldsymbol{x}, \tag{52}$$

where surface integrals are omitted.

It will be shown later that the Lagrangian L_A of (49) or (52) yields a rotational component of velocity field and in fact leads to a term generating non-zero helicity. In traditional formulations, this kind of Lagrangian is not considered. Then, in order to generate a rotational velocity field, an *ad hoc* term had to be introduced.

This section summarizes the above analysis on the Lagrangians of Eulerian space. In addition to the Lagrangian L_{T} defined in §2.2, one can add two Lagrangian functionals considered in §2.5:

$$L_\phi = -\int_M \mathrm{D}_t\phi \;\rho\, \mathrm{d}^3\boldsymbol{x}, \qquad L_\psi = -\int_M s\, \mathrm{D}_t\psi \;\rho\, \mathrm{d}^3\boldsymbol{x}, \tag{53}$$

and furthermore the one introduced in the previous section:

$$L_A = -\int_M \langle \mathcal{L}_W \boldsymbol{A},\, \boldsymbol{\omega}\,\rangle\, \mathrm{d}^3\boldsymbol{x} = \int_M \langle \boldsymbol{\Psi},\, \boldsymbol{\omega}\,\rangle\, \mathrm{d}^3\boldsymbol{x} = \int_M \langle \nabla\times\boldsymbol{\Psi},\, \boldsymbol{v}\,\rangle\, \mathrm{d}^3\boldsymbol{x} + \mathrm{Int}_S \tag{54}$$

$$= -\int_M \rho\,\langle \boldsymbol{b},\, \boldsymbol{v}\,\rangle\, \mathrm{d}^3\boldsymbol{x} + \mathrm{Int}_S = \int_M \langle \boldsymbol{A},\, \mathcal{L}_W^*[\boldsymbol{\omega}]\,\rangle\, \mathrm{d}^3\boldsymbol{x} + \mathrm{Int}_S. \tag{55}$$

Writing the total Lagrangian in the $\boldsymbol{x}$-space as

$$L_{\mathrm{T}}^* = \int_M \Lambda(\boldsymbol{v}, \rho, s, \phi, \psi, \boldsymbol{A})\; \mathrm{d}^3\boldsymbol{x},$$

its density Λ is given by

$$\Lambda \equiv \tfrac{1}{2}\rho\, v^k v^k - \rho\epsilon(\rho, s) - \rho \mathrm{D}_t\phi - \rho s\, \mathrm{D}_t\psi + \langle \boldsymbol{A},\, \mathcal{L}_W^*[\boldsymbol{\omega}]\,\rangle \tag{56}$$

$$= \tfrac{1}{2}\rho\,\langle \boldsymbol{v},\, \boldsymbol{v}\,\rangle - \rho\,\epsilon - \rho\,(\partial_t + \boldsymbol{v}\cdot\nabla)\phi - \rho\, s\,(\partial_t + \boldsymbol{v}\cdot\nabla)\psi + \langle \nabla\times\boldsymbol{\Psi},\, \boldsymbol{v}\,\rangle + \mathrm{Div}\,, \tag{57}$$

where Div is of divergence forms which are transformed to Int_S terms of L_{T}^*. This is the total Lagrangian density in the $\boldsymbol{x}$-space. Action I is defined by

$$I = \int \Lambda(\boldsymbol{v}, \rho, s, \phi, \psi, \boldsymbol{A})\; \mathrm{d}^4 x, \qquad \mathrm{d}^4 x = \mathrm{d}t\, \mathrm{d}^3\boldsymbol{x}.$$

Note: In classical mechanics of a point mass, the following is well-known. Under a Galilean boost transformation, $\boldsymbol{v} \to \boldsymbol{v} + \boldsymbol{U}$ (and $\boldsymbol{x} \to \boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{U}t$ with $\boldsymbol{U}$ a constant vector), the first term of Λ is *quasi-invariant*, because the factor $K = \frac{1}{2}\langle \boldsymbol{v},\, \boldsymbol{v}\,\rangle$ is transformed to

$$K' = K + \frac{\mathrm{d}}{\mathrm{d}t}\Delta, \qquad \Delta = \boldsymbol{X}(t, \boldsymbol{a})\cdot\boldsymbol{U} + \tfrac{1}{2}U^2 t. \tag{58}$$

The term $\mathrm{d}\Delta/\mathrm{d}t$ does not affect the Euler-Lagrange equation, as noted before. However, in quantum mechanics of Schrödinger equation for a wave function $\psi(t, \boldsymbol{x})$, invariance of the Schrödinger equation requires that ψ must be transformed by

$$\psi' = \exp\Big(\frac{i}{\hbar}\Delta(t, \boldsymbol{x})\Big)\,\psi(t, \boldsymbol{x}).$$

This states an explicit role played by the function $\Delta(t, \boldsymbol{x})$, yielding non-trivial cohomology of the Galilei group (Azcárraga & Izquierdo 1995, Ch.3).

If $K + \mathrm{d}\Delta/\mathrm{d}t$ is used in the Lagrangian in stead of K, then we have invariance with respect to the boost. The property of (58) will be reconsidered in §8.4 from a different point of view.

5 Variational principle in Eulerian space

5.1 *Variation of fields*

Variational principle requires that the action I is invariant with respect to variations of the fields of $\boldsymbol{v}, \rho, s, \phi, \psi$ and $\boldsymbol{A}$. By substituting varied fields $\boldsymbol{v} + \delta\boldsymbol{v},\ \rho + \delta\rho, \quad \cdots$ and $\boldsymbol{A} + \delta\boldsymbol{A}$ into $\Lambda(\boldsymbol{v}, \rho, s, \phi, \psi, \boldsymbol{A})$ and subtracting unvaried Λ, we obtain its variation $\delta\Lambda$, given by

$$\begin{aligned}
\delta\Lambda = \quad & \delta\boldsymbol{v} \cdot \rho\,(\,\boldsymbol{v} - \nabla\phi - s\,\nabla\psi - \boldsymbol{w}\,) \\
& + \delta\rho\,\left(\tfrac{1}{2}u^2 - h - \mathrm{D}_t\phi - s\,\mathrm{D}_t\psi - \boldsymbol{v}\cdot\boldsymbol{b}\,\right) \\
& - \delta s\ \rho\ \mathrm{D}_t\psi \\
& + \delta\phi\,\Big(\,\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v})\Big) - \partial_t(\rho\,\delta\phi) - \nabla\cdot(\rho\boldsymbol{v}\,\delta\phi) \\
& + \delta\psi\,\Big(\,\partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v})\,\Big) - \partial_t(\rho s\,\delta\psi) - \nabla\cdot(\rho s\boldsymbol{v}\,\delta\psi) \\
& + \langle\delta\boldsymbol{A},\ \mathcal{L}^*_W\boldsymbol{\omega}\rangle,
\end{aligned}$$

where the new term $\boldsymbol{w}$ is defined, with using (54), as

$$\frac{\delta L_A}{\delta v^k}\,\delta v^k = \langle\nabla\times\boldsymbol{\Psi},\,\delta\boldsymbol{v}\,\rangle \equiv -\rho\,\langle\boldsymbol{w},\,\delta\boldsymbol{v}\,\rangle, \tag{59}$$

Applying the principle of least action $\delta\,I = 0$ for arbitrary variations of $\delta\boldsymbol{v}$, $\delta\rho,\ \delta s,\ \delta\phi,\ \delta\psi$ and $\delta\boldsymbol{A}$, we obtain the followings:

$$\delta\boldsymbol{v}: \quad \boldsymbol{v} = \nabla\,\phi + s\,\nabla\psi + \boldsymbol{w}\,, \tag{60}$$
$$\delta\rho: \quad \tfrac{1}{2}v^2 - h - \mathrm{D}_t\phi - s\,\mathrm{D}_t\psi - \boldsymbol{v}\cdot\boldsymbol{b} = 0\,, \tag{61}$$
$$\delta s: \quad \mathrm{D}_t\psi \equiv \partial_t\psi + \boldsymbol{v}\cdot\nabla\psi = 0\,, \tag{62}$$
$$\delta\phi: \quad \partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0\,, \tag{63}$$
$$\delta\psi: \quad \partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v}) = 0\,, \tag{64}$$
$$\delta\boldsymbol{A}: \quad \mathcal{L}^*_W\boldsymbol{\omega} = 0\,. \tag{65}$$

Using the continuity equation (63), the equation (64) reduces to

$$\partial_t s + \boldsymbol{v} \cdot \nabla s = \mathrm{D}_t s = 0 \qquad \text{(adiabatic)}\,. \tag{66}$$

Thus, from the variational principle, the continuity equaiton (63) and the isentropic equation (66) have been derived. In most traditional formulations (Herivel 1955; Serrin 1959; Eckart 1960; Seliger & Witham 1968; Salmon 1988) of variational methods, these equations are imposed as constraints by using Lagrange multipliers.

From the Lagrangian L_A, three new results are derived: (i) the expression (60) of velocity $\boldsymbol{v}$ includes a new rotational term $\boldsymbol{w}$, (ii) the vorticity equation $\mathcal{L}^*_W[\boldsymbol{\omega}] = 0$ has been derived from the variation of $\boldsymbol{A}$, and (iii) the new term $\boldsymbol{w}$ leads to the source of helicity $\mathcal{H}$. The items (i) and (iii) are to be considered in §6.2 below.

Using (60) for the expression of velocity $\boldsymbol{v}$, the vorticity is given by

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \nabla \times (\nabla\,\phi + s\,\nabla\psi + \boldsymbol{w}) = \nabla s \times \nabla\psi + \nabla \times \boldsymbol{w}.$$

If the fluid is *homentropic*, and if the term $\boldsymbol{w}$ is omitted (equivalently if L_A is not taken into account), it is inevitable for us to obtain the solution $\boldsymbol{\omega} = 0$, that is an irrotational motion. It is generally understood that, even in such a homentropic fluid, it should be possible to have rotational flows. The Lagrangian L_A serves the need perfectly.

However, in the traditional approaches, the above mentioned property is thought as a defect of the formulation of Eulerian variation. In order to remedy this (apparent) flaw, Lin (1963) introduced conditions which keep identity of particles denoted by $\boldsymbol{a} = (a^k)$, with a Lagrangian represented as $\int A_k \cdot \mathrm{D}_t a^k \,\mathrm{d}^3\boldsymbol{x}$. This introduces three potentials $A_k(\boldsymbol{x}, t)$ as a set of Lagrange multipliers, resulting in a generalized expression of Clebsh representation (Kambe 2007; Clebsch 1859). But, physical significance of the potentials is not clear (Seliger & Witham 1968; Bretherton 1970; Salmon 1988).

From the invariance properties in the $\boldsymbol{a}$-space, corresponding equations in the $\boldsymbol{x}$-space have been derived as follows:

$$\partial_\tau(\mathrm{d}^3\boldsymbol{a}) = 0\ : \qquad \partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0\,, \tag{67}$$

$$\partial_\tau\, s = 0\ : \qquad \partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v}) = 0\,, \tag{68}$$

$$\partial_\tau\, \boldsymbol{\Omega}_a = 0\ : \qquad \partial_t\boldsymbol{\omega} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) = 0. \tag{69}$$

Invriance of the action with respect to variations of particle coordinates in the $\boldsymbol{x}$-space leads to the equation of momentum conservation. Suppose that the particle coordinates $\boldsymbol{x} = \boldsymbol{X}(t, \boldsymbol{a})$ are transformed infinitesimally to $\boldsymbol{x}'$ as follows:

$$\boldsymbol{x}'(\boldsymbol{x}, t) = \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t), \tag{70}$$

where the coordinates of a particle $\boldsymbol{a}$ are transformed from $\boldsymbol{X}(t, \boldsymbol{a})$ to $\boldsymbol{X}' = \boldsymbol{X}(t, \boldsymbol{a}) + \boldsymbol{\xi}(\boldsymbol{X}, t)$. It is required that the action I is invariant in this transformation. What is changed with this transformation is only the *expression* of Eulerian coordinates of particles. This is regarded as a *gauge* transformation in the Eulerian space.

The invariance equations (1) and (3) are assumed implicitly. A volume element $\mathrm{d}^3\boldsymbol{x}$ is transformed to $\mathrm{d}^3\boldsymbol{x}' = (1 + \partial_k \xi^k)\mathrm{d}^3\boldsymbol{x}$ up to linear terms. Hence the volume variation is given by $\Delta(\mathrm{d}^3\boldsymbol{x}) = \partial_k \xi^k \, \mathrm{d}^3\boldsymbol{x}$. On the other hand, variations of density and entropy are given by $\Delta\rho = -\rho \, \partial_k \xi^k$ and $\Delta s = 0$ respectively. Variation of velocity is $\Delta\boldsymbol{v} = \mathrm{D}_t \boldsymbol{\xi}$ from (39).

Taking the variation (70) under the assumption that ϕ, ψ and $\boldsymbol{\Psi}$ are fixed, the resulting variation of I is given by

$$\Delta I = \int \mathrm{d}^4 x \left[\frac{\partial \Lambda}{\partial \boldsymbol{v}} \Delta \boldsymbol{v} + \frac{\partial \Lambda}{\partial \rho} \Delta \rho + \frac{\partial \Lambda}{\partial s} \Delta s + \Lambda \, \partial_k \xi^k \right].$$

It is required that this vanishes for any arbitrary variation ξ^k. Substtuting the expressions of $\Delta\boldsymbol{v}$, $\Delta\rho$ and Δs mentioned above, we obtain the following equation (after integrations by parts):

$$\frac{\partial}{\partial t}\Big(\frac{\partial \Lambda}{\partial v^k}\Big) + \frac{\partial}{\partial x^l}\Big(v^l \, \frac{\partial \Lambda}{\partial v^k}\Big) + \frac{\partial}{\partial x^k}\Big(\Lambda - \rho \, \frac{\partial \Lambda}{\partial \rho}\Big) = 0. \tag{71}$$

Similarly, for any arbitrary variations $\Delta\phi$ and $\Delta\psi$, the invariance of I leads to the continuity equaiton and the isentropic equation, already given by (63) and (66) respectively.

Defining momentum density m_k and momentum flux tensor M_k^l by

$$m_k = \frac{\partial \Lambda}{\partial v^k}, \qquad M_k^l = v^l \, \frac{\partial \Lambda}{\partial v^k} + \Big(\Lambda - \rho \, \frac{\partial \Lambda}{\partial \rho}\Big) \, \delta_k^l \,, \tag{72}$$

the equation (71) reduces to the following conservation equation of momentum:

$$\partial_t \, m_k + \partial_l \, M_k^l = 0 \,. \tag{73}$$

We have $m_k = \rho v_k$ and $M_k^l = \rho v_k v^l + p\,\delta_k^l$ from (56) ($v_k = v^k$ in the present case of Euclidean space). Substituting these, we obtain the following conservation equation for fluid momentum:

$$\partial_t\Big(\rho v^k\Big) + \partial_l\Big(\rho v^l v^k\Big) + \partial_k p = 0\,. \tag{74}$$

Using (63), this reduces to Euler's equation of motion:

$$\partial_t v^k + (v^l\partial_l)v^k = -\frac{1}{\rho}\,\partial_k\,p \quad (= -\partial_k\,h). \tag{75}$$

This is equivalent to (18).

Energy equation can be derived by a combination of the above equations. Taking inner product of (75) with ρv^k,

$$\rho\,\partial_t\frac{v^2}{2} + \rho v^l\partial_l\frac{v^2}{2} + v^k\partial_k\,p = 0.$$

Adding the continuity equation (63) multiplied by $v^2/2$ to this equation, and using $\mathrm{d}p = \rho\,\mathrm{d}h$ from (4), we obtain

$$\partial_t\left(\rho\frac{v^2}{2}\right) + \partial_k\left(\rho v^k\frac{v^2}{2}\right) + \rho v^k\partial_k\,h = 0. \tag{76}$$

From the first of (4) and using the definition $h = \epsilon + p/\rho$,

$$\partial_t(\rho\epsilon) - h\,\partial_t\rho = \partial_t(\rho\epsilon) + h\,\partial_k(\rho v^k) = 0.$$

Adding this to (76), we obtain the following conservation equation of energy:

$$\partial_t\left[\rho(\tfrac{1}{2}\,v^2 + \epsilon)\right] + \partial_k\left[\rho v^k\,(\tfrac{1}{2}\,v^2 + h)\right] = 0. \tag{77}$$

This is equivalent to (15) (Kambe 2008a).

6 Invariants: L_A and Helicity

We consider in this section that the Lagrangian L_A and helicity $\mathcal{H}$ are in fact invariants, and that both of them are related to each other.

Using the one-form $\mathcal{V}^1$ and two-form $\Omega^2 = \mathrm{d}\mathcal{V}^1$ defined in §3.3, one can define the following three-form H^3 on a three-dimensional manifold:

$$H^3 = \mathcal{V}^1 \wedge \Omega^2,$$

Let us take differntial of H^3. Since $\mathrm{d}\mathcal{V}^1 = \Omega^2$ and $\mathrm{d}\Omega^2 = \mathrm{d}^2\mathcal{V}^1 = 0$, we have

$$\mathrm{d}H^3 = \mathrm{d}\mathcal{V}^1 \wedge \Omega^2 - \mathcal{V}^1 \wedge \mathrm{d}\Omega^2 = \Omega^2 \wedge \Omega^2 = 0, \tag{78}$$

since the manifold is three-dimensional. Namely, H^3 is a closed three-form. Hence we have *Cohomology* classes of H^3.

In fact, replacing A^1 of (34) and (41) with $\mathcal{V}^1$ (defined by (28) and (29)), we obtain

$$H^3 = \mathcal{V}^1 \wedge \Omega^2 = \langle \boldsymbol{V}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3\boldsymbol{a}, \quad = \langle \boldsymbol{v},\, \boldsymbol{\omega}\, \rangle\, \mathrm{d}^3\boldsymbol{x}. \tag{79}$$

This H^3 may be called as a helicity three-form. Operating ∂_τ and using (32) and (33), we have

$$\partial_\tau H^3 = \partial_\tau \mathcal{V}^1 \wedge \Omega^2 - \mathcal{V}^1 \wedge \partial_\tau \Omega^2 = -\mathrm{d}F^0 \wedge \Omega^2 = -\mathrm{d}(F^0 \Omega^2). \tag{80}$$

(F^0 is a zero-form defined by $-\frac{1}{2}\, v^2 + h$.) Helicity $\mathcal{H}$ is defined by integrating (79):

$$\mathcal{H}[\mathcal{V}^1, \mathrm{d}\mathcal{V}^1] \equiv \int H^3 = \int_{M(\boldsymbol{a})} \langle \boldsymbol{V}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3\boldsymbol{a} = \int_{M(\boldsymbol{x})} \langle \boldsymbol{v},\, \boldsymbol{\omega} \rangle\, \mathrm{d}^3\boldsymbol{x} = \mathcal{H}[\boldsymbol{v}, \boldsymbol{\omega}].$$

Differentiating this with respect to τ and using (80),

$$\partial_\tau \mathcal{H} = \int \partial_\tau H^3 = -\int_M \mathrm{d}(F^0 \Omega^2) = -\int_{\partial M} F^0 \Omega^2.$$

The right hand side is a surface integral over ∂M bounding M, and vanishes in the following cases: (a) M is unbounded, and $\boldsymbol{v}$ and $\boldsymbol{\omega}$ decay sufficiently rapidly, or (b) M is bounded with peiodic boundary conditions. In these cases, $\mathcal{H}$ is a time invariant.

The equations (78) and (80) are equivalent to the conditions for the Secondary invariant of Chern-Simons (Chern 1979). $\mathcal{H}[\boldsymbol{v}, \boldsymbol{\omega}]$ is a topological invariant, and a set of $\mathcal{H}[\boldsymbol{v}, \boldsymbol{\omega}]$ constitutes a class of cohomology.

6.2 Source of helicity

Present formulation enables us to identify the helicity source. Using (48), $-\boldsymbol{\Psi}$ can be written as

$$-\boldsymbol{\Psi} = \mathcal{L}_W \boldsymbol{A} = \partial_t \boldsymbol{A} + (\nabla \times \boldsymbol{A}) \times \boldsymbol{v} + \nabla(A_k v^k).$$

By defining a vector $\boldsymbol{B}$ by $\nabla \times \boldsymbol{A}$, we have

$$\begin{aligned} \boldsymbol{B} &= \nabla \times \boldsymbol{A}, \\ -\nabla \times \boldsymbol{\Psi} &= \partial_t \boldsymbol{B} + \nabla \times (\boldsymbol{B} \times \boldsymbol{v}) \equiv \mathcal{L}^*_W[\boldsymbol{B}]. \end{aligned} \tag{81}$$

These are written in the following differential forms respectively:

$$B^2 = \mathrm{d}A^1 = \boldsymbol{B} \cdot \boldsymbol{s}^2 = \boldsymbol{B}_a \cdot \boldsymbol{S}^2, \tag{82}$$

$$-\mathrm{d}\Psi^1 = \partial_\tau B^2 = \mathcal{L}_W B^2 = \mathcal{L}^*_W[\boldsymbol{B}] \cdot \boldsymbol{s}^2, \tag{83}$$

where the notations of (30) and (31) are used for $\boldsymbol{S}^2$ and $\boldsymbol{s}^2$, and Ψ^1 is defined in (47). The right hand side of (83) is obtained from (43) by repacing Ω^2 with B^2.

In order to obtain an explicit expression of $\boldsymbol{w}$, we substitute the variation $\boldsymbol{v} + \delta \boldsymbol{v}$ into $\boldsymbol{v}$ of L_A represented by (54) where $\boldsymbol{\Psi}$ is kept fixed. From (59), the variation of L_A is

$$\delta L_A = \int_M \langle \nabla \times \boldsymbol{\Psi}, \delta \boldsymbol{v} \rangle \, \mathrm{d}^3\boldsymbol{x} = -\int_M \rho \, \langle \boldsymbol{w}, \delta \boldsymbol{v} \rangle \, \mathrm{d}^3\boldsymbol{x}.$$

Therefore, $\boldsymbol{w}$ is given by

$$\rho \boldsymbol{w} = -\nabla \times \boldsymbol{\Psi} = \mathcal{L}^*_W[\boldsymbol{B}]. \tag{84}$$

Total velocity defined by (60) is

$$\boldsymbol{v} = \nabla \phi + s \nabla \psi + \boldsymbol{w}, \qquad \boldsymbol{w} = -\frac{1}{\rho} \nabla \times \boldsymbol{\Psi}. \tag{85}$$

Suppose that the entropy takes a uniform value $s = s_0$. Then we have $\boldsymbol{v} = \nabla \Phi + \boldsymbol{w}$ (where $\Phi = \phi + s_0 \psi$), and the vorticity is given by

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{w}. \tag{86}$$

In this case, the helicity is

$$\mathcal{H} = \int_V \langle \boldsymbol{\omega}, \boldsymbol{v} \rangle \, \mathrm{d}^3\boldsymbol{x} = \int_V \langle \boldsymbol{\omega}, \boldsymbol{w} \rangle \, \mathrm{d}^3\boldsymbol{x} = \int_V \rho^{-1} \langle \boldsymbol{\omega}, \mathcal{L}^*_W[\boldsymbol{B}] \rangle \, \mathrm{d}^3\boldsymbol{x}. \tag{87}$$

Here, the term $\boldsymbol{\omega}\cdot\nabla\Phi = \nabla\cdot(\Phi\boldsymbol{\omega})$ is omitted since it is transformed to a vanishing surface integral in the cases (a) and (b) of §6.1. The above integral (87) expresses that the inner product of $\boldsymbol{\omega}$ with

$$\mathcal{L}_W^*[\boldsymbol{B}] = \partial_t \boldsymbol{B} + \nabla\times(\boldsymbol{B}\times\boldsymbol{v}) \qquad (\boldsymbol{B} = \nabla\times A) \tag{88}$$

(devided by ρ) gives the helicity source.

6.3 Case of an incompressible fluid

If the fluid is incompressible ($\rho = \rho_0 = const$), and in addition if the entropy is uniform ($s = s_0 = const$), the velocity $\boldsymbol{v}$ has a scalar potential Φ and a vector potential $\boldsymbol{\Psi}_*$. In fact from (85), we have

$$\boldsymbol{v} = \nabla\,\Phi + \nabla\times\boldsymbol{\Psi}_* = \nabla\,\Phi + \boldsymbol{w}_*, \tag{89}$$

where $\Phi = \phi + s_0\psi$, and $\boldsymbol{\Psi}_* = -\boldsymbol{\Psi}/\rho_0$, and the rotational component $\boldsymbol{w}_*$ is represented by

$$\boldsymbol{w}_* = \nabla\times\boldsymbol{\Psi}_* = \frac{1}{\rho_0}\,\mathcal{L}_W^*[\boldsymbol{B}], \tag{90}$$

from (84). In order to rewrite the helicity, we define one-form W^1 dual to the velocity $\boldsymbol{w}_*$ by

$$\mathcal{W}^1 = \boldsymbol{w}_*\cdot \mathrm{d}\boldsymbol{x} = \frac{1}{\rho_0}\,\mathcal{L}_W^*[\boldsymbol{B}]\cdot\mathrm{d}\boldsymbol{x}, \tag{91}$$

according to the notations of (28) and (29), and (90). We have the equality:

$$\mathcal{L}_W^*[\boldsymbol{B}]\cdot\mathrm{d}\boldsymbol{x} = \partial_\tau \boldsymbol{B}_a\cdot\mathrm{d}\boldsymbol{a} = \partial_\tau B^1, \qquad B^1 \equiv \boldsymbol{B}_a\cdot\mathrm{d}\boldsymbol{a}, \tag{92}$$

which is implied by (82) and (83). Then the helicity is given by

$$\mathcal{H} = \int \mathcal{W}^1\wedge\Omega^2 = \frac{1}{\rho_0}\int_V \langle\mathcal{L}_W^*[\boldsymbol{B}],\,\boldsymbol{\omega}\rangle\,\mathrm{d}^3\boldsymbol{x} = \frac{1}{\rho_0}\int_V \langle\partial_\tau\boldsymbol{B}_a,\,\boldsymbol{\Omega}_a\rangle\,\mathrm{d}^3\boldsymbol{a}, \tag{93}$$

($\boldsymbol{\omega} = \nabla\times\boldsymbol{w}_*$), where $\mathcal{V}^1$ is replaced by $\mathcal{W}^1$ since the potential part $\nabla\,\Phi$ gives vanishing contribution to $\mathcal{H}$.

The Lagrangian $L_A = -\int_M\langle\partial_\tau\boldsymbol{A}_a,\boldsymbol{\Omega}_a\rangle\mathrm{d}^3\boldsymbol{a}$ is written in the $\boldsymbol{x}$-space variables by (49), which is rewritten again as

$$L_A = -\rho_0\int_M\langle\boldsymbol{\Psi}_*,\,(\nabla\times\boldsymbol{w}_*)\,\rangle\,\mathrm{d}^3\boldsymbol{x} = -\rho_0\int_M\langle\boldsymbol{w}_*,\ \boldsymbol{w}_*\,\rangle\,\mathrm{d}^3\boldsymbol{x} + \mathrm{Int}_S, \tag{94}$$

by integration by parts. It is found that $-\frac{1}{2}L_A$ is equivalent to the kinetic energy of rotational component $\boldsymbol{w}_*$, apart from the surface integral term Int_S.

6.4 Electrromagnetic analogy

In order to seek a relation between $\mathcal{H}$ and L_A, we define one-form $\mathcal{A}^1$ on a four-dimensional manifold $(\tau, \boldsymbol{a})$:

$$\mathcal{A}^1 = \Phi\, \mathrm{d}\tau + A^1, \qquad A^1 = A_a\, \mathrm{d}a + A_b\, \mathrm{d}b + A_c\, \mathrm{d}c.$$

Applying the exterior derivative $\mathbf{d}_4$ defined by (35), we obtain a two-form $\mathcal{F}^2$ as

$$\begin{aligned}\mathcal{F}^2 = \mathbf{d}_4 \mathcal{A}^1 &= \mathrm{d}\Phi \wedge \mathrm{d}\tau + \mathrm{d}\tau \wedge \partial_\tau A^1 + \mathrm{d}A^1 \\ &= E^1 \wedge \mathrm{d}\tau + B^2.\end{aligned}$$

where

$$E^1 = -\partial_\tau A^1 + \mathrm{d}\Phi \quad \Rightarrow \quad \boldsymbol{E}_a = -\partial_\tau \boldsymbol{A}_a + \nabla_a \Phi, \tag{95}$$
$$B^2 = \mathrm{d}A^1 \quad \Rightarrow \quad \boldsymbol{B}_a = \nabla_a \times \boldsymbol{A}_a. \tag{96}$$

The vector equations on the right of the arrow "$\Rightarrow$" are equivalent expressions derived from the equations of differential forms on its left, where $\boldsymbol{A}_a$ is defined in §3.4.

Operating $\mathbf{d}_4$ on $\mathcal{F}^2$ and using $\mathbf{d}_4 \mathcal{F}^2 = \mathbf{d}_4^2 \mathcal{A}^1 = 0$, we obtain

$$0 = \mathbf{d}_4 \mathcal{F}^2 = (\mathrm{d}E^1 + \partial_\tau B^2) \wedge \mathrm{d}\tau + \mathrm{d}B^2. \tag{97}$$

Since $\mathrm{d}B^2 = \mathrm{d}^2 A^1 = 0$, we find the following equation:

$$\mathrm{d}E^1 + \partial_\tau B^2 = 0 \quad \Rightarrow \quad \nabla_a \times \boldsymbol{E}_a + \partial_\tau \boldsymbol{B}_a = 0. \tag{98}$$

This is equivalent to the form of Faraday's law in the electromagnetism. Substituting the expression (95) for $\boldsymbol{E}_a$ into the vector equation on the right, we obtain $\partial_\tau \boldsymbol{B}_a = \nabla_a \times \partial_\tau \boldsymbol{A}_a$.

The helicity of (93) is rewritten by using (98) as follows:

$$\mathcal{H} = \frac{1}{\rho_0} \int_V \langle \partial_\tau \boldsymbol{B}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3 \boldsymbol{a} = -\frac{1}{\rho_0} \int_V \langle \nabla_a \times \boldsymbol{E}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3 \boldsymbol{a}. \tag{99}$$

On the other hand, using the relation $\partial_\tau \boldsymbol{A}_a = -\boldsymbol{E}_a + \nabla_a \Phi$ obtained from (95), the Lagrangian L_A of (38) can be rewritten as

$$L_A = -\int_M \langle \partial_\tau \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3 \boldsymbol{a} = \int_M \langle \boldsymbol{E}_a - \nabla_a \Phi,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3 \boldsymbol{a} = \int_M \langle \boldsymbol{E}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3 \boldsymbol{a}, \tag{100}$$

where the term $\boldsymbol{\Omega}_a \cdot \nabla_a \Phi = \nabla_a \cdot (\Phi \boldsymbol{\Omega}_a)$ is neglected, because it is transformed to surface integrals.

Now, it has become clear that the helicity $\mathcal{H}$ and the Lagrangian L_A are not independent to each other, but that there is a relation between them. The latter depends on the field $\boldsymbol{E}_a$ (an electric-like field) linearaly, whereas the former depends on $\partial_\tau \boldsymbol{B}_a$ linearaly ($\boldsymbol{B}_a$: a magnetic-like field). The two fields $\boldsymbol{E}_a$ and $\boldsymbol{B}_a$ are related by (98). Therefore, if one of them exists, the other also exists. In the traditional fluid mechanics, although the helicity $\mathcal{H}$ was investigated in detail, the Lagrangian L_A has never been considered explicitly.

6.5 Time invariance of $\mathcal{H}$ and L_A

Assuming that the fluid is incompressible and homentropic, the one-form $\mathcal{V}^1$ is

$$\mathcal{V}^1 = \mathrm{d}_a \Phi + \mathcal{W}^1 = \mathrm{d}_a \Phi + \frac{1}{\rho_0}\, \partial_\tau B^1,$$

from (89), (91) and (92), where $\mathrm{d}_a \Phi = \nabla_a \Phi \cdot \mathrm{d}\boldsymbol{a}$. From this, we have

$$\Omega^2 = \mathrm{d}\mathcal{V}^1 = \mathrm{d}\mathcal{W}^1 = \frac{1}{\rho_0}\, \partial_\tau (\mathrm{d}B^1). \tag{101}$$

Since the vorticity two-form $\Omega^2 = \mathrm{d}\mathcal{V}^1$ is time invariant, the above can be integrated immediately, and we obtain the following for $\mathrm{d}B^1$:

$$\mathrm{d}B^1 = \rho_0\, \Omega^2\, \tau \quad \Rightarrow \quad \nabla_a \times \boldsymbol{B}_a = \rho_0\, \tau\, \nabla_a \times \boldsymbol{W}_a(\boldsymbol{a}), \tag{102}$$

where $\boldsymbol{W}_a$ is a vector dual to $\mathcal{W}^1 = \boldsymbol{W}_a \cdot \mathrm{d}\boldsymbol{a}$, which should be τ-independent because $\Omega^2 = \mathrm{d}\mathcal{W}^1$ is so. Solving the vector equation of (102), we obtain

$$\boldsymbol{B}_a = \rho_0 \tau\, \boldsymbol{W}_a(\boldsymbol{a}), \tag{103}$$

where an inessential gradient term is dropped on the right hand side. Thus, we find

$$\partial_\tau \boldsymbol{B}_a = \rho_0\, \boldsymbol{W}_a(\boldsymbol{a}). \tag{104}$$

In view of the relation $\boldsymbol{B}_a = \nabla_a \times \boldsymbol{A}_a$, we have

$$\partial_\tau \boldsymbol{A}_a = \rho_0\, \boldsymbol{\alpha}(\boldsymbol{a}) + \nabla_a \phi_a, \qquad \nabla_a \times \boldsymbol{\alpha} = \boldsymbol{W}_a. \tag{105}$$

where $\boldsymbol{\alpha}(\boldsymbol{a})$ is τ-independent, and $\phi_a(\boldsymbol{a})$ a scalar function.

Substituting (104) into (99),

$$\mathcal{H} = \int_V \langle \boldsymbol{W}_a(\boldsymbol{a}),\, \boldsymbol{\Omega}_a(\boldsymbol{a}) \rangle \, \mathrm{d}^3\boldsymbol{a}. \tag{106}$$

Substituting (105) into (100),

$$L_A = -\int_M \langle \partial_\tau \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle \, \mathrm{d}^3\boldsymbol{a} = -\rho_0 \int_M \langle \boldsymbol{\alpha}(\boldsymbol{a}),\, \boldsymbol{\Omega}_a(\boldsymbol{a}) \rangle \, \mathrm{d}^3\boldsymbol{a}. \tag{107}$$

Thus, it is found that $\mathcal{H}$ and L_A are τ-independent constants.

7 Uniqueness of transformation

Local transformation from the Lagrangian $\boldsymbol{a}$-space to the Eulerian $\boldsymbol{x}$-space is determined uniquely by nine components of the 3×3 matrix $\partial x^k/\partial a^l$, as noted in §2.4. However, the solutions given in the beginning at §2.3, we had six relations including the nine components. Namely, we had three relations of (12) between $\boldsymbol{v} = (X_\tau, Y_\tau, Z_\tau)$ and (V_a, V_b, V_c), and another three relations of (16) between $\mathcal{A} = (X_{\tau\tau}, Y_{\tau\tau}, Z_{\tau\tau})$ and $(\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c)$.

In §2.3, vorticity field was not taken into consideration in the Euler-Lagrange equations: (11) and (15). This is because fluid particles are fixed by definition in the (a, b, c) coordinate space. However in the (x, y, z) space, there is local rotation of fluid particles. Transformation between the two spaces must take into account the local rotation.

Remaining three relations are given by the equations connecting the $\boldsymbol{a}$-space vorticity, $\boldsymbol{\Omega}_a(\boldsymbol{a}) = (\Omega_a,\, \Omega_b, \Omega_c)$ of (30), with the $\boldsymbol{x}$-space vorticity, $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ of (31). For example, Ω_a is given by the following:

$$\begin{aligned}\Omega_a = {} & \omega_x\,(\partial_b y\,\partial_c z - \partial_c y\,\partial_b z) + \omega_y\,(\partial_b z\,\partial_c x - \partial_c z\,\partial_b x) \\ & + \omega_z\,(\partial_b x\,\partial_c y - \partial_c x\,\partial_b y).\end{aligned} \tag{108}$$

There is a set of three vectors (velocity, acceleration and vorticity) in each of $\boldsymbol{a}$-space and $\boldsymbol{x}$-space, which are determined by evolution equations subject to given initial conditions in each space. For each of the three vectors, there are three transformation relations at each point between components of each space. These nine equations are necessary and sufficient to determine nine matrix elements $\partial x^k/\partial a^l$ locally. Thus, the transformation between the $\boldsymbol{a}$-space and the $\boldsymbol{x}$-space is determined uniquely (Kambe 2008a).

8 Gauge symmetries (Translation symmetry and rotation symmetry)

First, Lagarangian density Λ defined by (56) or (57) are reproduced here:

$$\Lambda \equiv \tfrac{1}{2}\rho \langle \boldsymbol{v}, \boldsymbol{v}\rangle - \rho\epsilon(\rho, s) - \rho \mathrm{D}_t\phi - \rho s\, \mathrm{D}_t\psi - \langle \mathcal{L}_W \boldsymbol{A},\, \boldsymbol{\omega}\,\rangle. \tag{109}$$

Translation symmetry is investigated first in the sections 8.1 to 8.4. Then, rotation symmetry is considered in §8.5 and 8.6. Our target is to verify the invariance property of Λ with respect to the transformations of both translation and rotation.

8.1 Local gauge transformation (translation)

Suppose that there are two Eulerican coordinate frames F and F'. We consider a transformation of the position coordinate $\boldsymbol{x}$ of F to $\boldsymbol{x}'$ of another frame F'. Suppose that the transformation is given by

$$LGT: \qquad \boldsymbol{x}'(\boldsymbol{x}, t) = \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t), \quad t' = t. \tag{110}$$

This is regarded as a local gauge transformation LGT between two non-inertial frames of reference F and F'. In fact, it means that the position coordinate $\boldsymbol{x}$ of a fluid particle in the frame F is transformed to a new position coordinate $\boldsymbol{x}'$ given by $X'_a(X_a, t) = X_a(t) + \boldsymbol{\xi}(X_a, t)$ in the frame F'. Therefore, its velocity $\boldsymbol{v} = (\mathrm{d}/\mathrm{d}t)X_a(t)$ is transformed to

$$\boldsymbol{v}'(\boldsymbol{x}') \equiv \frac{\mathrm{d}}{\mathrm{d}t'}X'_a = \frac{\mathrm{d}}{\mathrm{d}t}\Big(X_a(t) + \boldsymbol{\xi}(X_a, t)\Big) = \boldsymbol{v}(X_a) + (\mathrm{d}/\mathrm{d}t)\boldsymbol{\xi}_a, \tag{111}$$

$$(\mathrm{d}/\mathrm{d}t)\boldsymbol{\xi}_a = \partial_t \boldsymbol{\xi} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\xi}\,\Big|_{\boldsymbol{x}=X_a}, \qquad \boldsymbol{\xi}_a \equiv \boldsymbol{\xi}(X_a, t). \tag{112}$$

This is a transformation between two coordinate values of the *same* particle described by two different frames of reference F and F'. Physically, two vectors $\boldsymbol{x}$ and $\boldsymbol{x}'$ denote the same point, represented by the same value of Lagrange coordinte $\boldsymbol{a}$. Namely, we are considering a gauge transformation between two reference frames.

According to the transformation (110), the time derivative ∂_t and space derivative $\partial_k = \partial/\partial x^k$ in the frame F are related to the derivatives $\partial_{t'}$ and $\partial'_k = \partial/\partial x'_k$ of F' as follows:

$$\partial_t = \partial_{t'} + (\partial_t \boldsymbol{\xi}) \cdot \nabla', \qquad \nabla' = (\partial'_k), \tag{113}$$

$$\partial_k = \partial'_k + \ \partial_k \xi_l \ \partial'_l, \qquad \partial'_k = \partial/\partial x'_k \, . \tag{114}$$

The transformation LGT of (110) is also called a local *translational* transformation.

8.2 Gauge invariance of D_t

The operator $\mathrm{D}_t \equiv \partial_t + (\boldsymbol{v} \cdot \boldsymbol{\nabla})$ is invariant with respect to LGT: *i.e.* $\mathrm{D}_t = \mathrm{D}'_t$. In fact from (111) and (114), we have

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} = \boldsymbol{v} \cdot \boldsymbol{\nabla}' + (\boldsymbol{v} \cdot \boldsymbol{\nabla} \xi) \cdot \boldsymbol{\nabla}' = \boldsymbol{v}'(\boldsymbol{x}') \cdot \boldsymbol{\nabla}' + \Big(- (\mathrm{d}\boldsymbol{\xi}/\mathrm{d}t) + \boldsymbol{v} \cdot \boldsymbol{\nabla} \xi \Big) \cdot \nabla',$$

where $\boldsymbol{v} = \boldsymbol{v}' - \mathrm{d}\boldsymbol{\xi}/\mathrm{d}t$ is used. The last term is rewritten as

$$\Big(- (\mathrm{d}\boldsymbol{\xi}/\mathrm{d}t) + \boldsymbol{v} \cdot \boldsymbol{\nabla} \xi \Big) \cdot \nabla' \ = \ -\partial_t \boldsymbol{\xi} \cdot \nabla' = \partial_{t'} - \partial_t,$$

by using (112) and (113). Hence, we have

$$\partial_t + \boldsymbol{v} \cdot \boldsymbol{\nabla} = \partial_{t'} + \boldsymbol{v}' \cdot \boldsymbol{\nabla}'. \tag{115}$$

This means that the operator D_t satisfies the invariance with respect to local translational transformations, *i.e.* the translation symmetry. Thus, the operator D_t is the *covariant derivative* in the sense of gauge theory (Weinberg 1995; Kambe 2007).

The particle labels $a^i(t, \boldsymbol{x})$ $(i = 1, 2, 3)$ are scalars, and move together with the material particle with the velocity $\boldsymbol{v} = \partial_t X(t, \boldsymbol{a})$. Hence, we have

$$\mathrm{D}_t a^i = \partial_t a^i + (\boldsymbol{v} \cdot \nabla) a^i = 0,$$

always. Writing the particle position as $\boldsymbol{x} = X(t, \boldsymbol{a})$, we have

$$\mathrm{D}_t \boldsymbol{x} = \mathrm{D}_t X(t, \boldsymbol{a}(t, \boldsymbol{x})) = \partial_t X(t, \boldsymbol{a}) + \mathrm{D}_t \boldsymbol{a} \cdot \nabla_a X = \partial_\tau X(\tau, \boldsymbol{a}) = \boldsymbol{v}. \tag{116}$$

where $(\nabla_a \boldsymbol{X}) = (\partial X^k / \partial a^l)$. Thus, we have the equality: $\partial_\tau = \mathrm{D}_t = \partial_t + (\boldsymbol{v} \cdot \boldsymbol{\nabla})$, which was used already.

Velocity $\boldsymbol{v}$ can be defined by $\mathrm{D}_t \boldsymbol{x}$. In fact, operating $\mathrm{D}'_{t'}$ on the equation (110) and using $\mathrm{D}'_{t'} = \mathrm{D}_t$, we obtain

$$\boldsymbol{v}' = \mathrm{D}'_{t'} \boldsymbol{x}' = \mathrm{D}_t (\boldsymbol{x} + \boldsymbol{\xi}) = \boldsymbol{v} + \mathrm{D}_t \boldsymbol{\xi}. \tag{117}$$

This coincides with (111). Thus, the particle velocity is defined by $\boldsymbol{v}(\boldsymbol{x}, t) = \mathrm{D}_t \boldsymbol{x}$.

8.3 Gauge transformation (a general formulation)

Suppose that we have a group $\mathcal{G}$, and consider the following transformation by its element $g \in \mathcal{G}$.

(***a***) A field $u(x)$ is defined on a manifold M. Suppose that the coordinate $x \in M$ is transformed to $x' = gx$ by $g \in \mathcal{G}$, and the field u to u' defined by $u'g = gu$ simultaneously. Then we have

$$u'(x') \equiv u'g\,x \;= gu\,x = gu(x) \tag{118}$$

This means that u is transformed in the same way as the coordinate x.

(***b***) Next, suppose that a field of group element $g(x)$ (where $g \in \mathcal{G}$) is defined at each point $x \in M$, and $u(x)$ is transformed according to $u'g = gu$.

In addition, in place of the partial derivative ∂_t (with respect to time), we define a covariant derivative $\mathrm{D}_t = \partial_t + A$ by introducing a gauge field A. Its transformation is assumed to be given by

$$\mathrm{D}'_{t'}\,g = g\,\mathrm{D}_t,$$

where $\mathrm{D}'_{t'} = \partial' + A'$ and $\partial' = \partial/\partial t'$. Operating the left side on u, we obtain $\mathrm{D}'_{t'}\,g\,u = \mathrm{D}'_{t'}\,u'g$. Equating this to the right side, we have

$$\mathrm{D}'_{t'}\,u'g \equiv (\mathrm{D}_t u)'g = g\,\mathrm{D}_t u. \tag{119}$$

This means that $\mathrm{D}_t u$ is transformed with the same way as u.

In the example of the previous section where g is LGT, by setting u to be the particle coordinate $\boldsymbol{x}$, $\mathrm{D}_t u$ corresponds to the velocity $\boldsymbol{v}$, and the equation (117) can be written as

$$\boldsymbol{v}'(\boldsymbol{x}') = \mathrm{D}'_{t'}\boldsymbol{x}' = g\,\mathrm{D}_t\boldsymbol{x} = g\,\boldsymbol{v}(\boldsymbol{x}),$$

where $g\boldsymbol{v}$ is defined by $\boldsymbol{v} + \mathrm{D}_t\boldsymbol{\xi}$. We consider this kind of transformations below.

8.4 Translation symmetry

Aim of this subsection is to verify that Λ is invariant with respect to LGT where the coordinate $\boldsymbol{x}$ transforms according to (110), and the velocity $\boldsymbol{v}$ according to (117). In regard to the first term $\frac{1}{2}\rho\langle\boldsymbol{v},\,\boldsymbol{v}\rangle$, a special consideration is necessary as noted in §4.3. A Galilean transformation between two frames of reference in relative motion with a constant velocity $\boldsymbol{U}$ is given by $\boldsymbol{\xi} = \boldsymbol{U}t$ and $\boldsymbol{v}' = \boldsymbol{v}+\boldsymbol{U}$. Substituting the latter in $\langle \boldsymbol{v}', \boldsymbol{v}' \rangle$, we have $\langle \boldsymbol{v}, \boldsymbol{v} \rangle + 2\langle \boldsymbol{v}, \boldsymbol{U} \rangle + \langle \boldsymbol{U}, \boldsymbol{U} \rangle$, in which no invariance is seen. This problem is to be resolved by a relativistic Lorentz transformation, as follows.

Galilean transformation is regarded as a limiting one of the Lorentz transformation of space-time $(x^\mu) = (t, \boldsymbol{x})$ as $v/c \to 0$ (c: light speed). In the limit of $v/c \to 0$, a Lorentz invariant Lagrangian takes the following form denoted by $\Lambda_{\rm L}^{(0)}$ (Landau & Lifshitz 1987):

$$\Lambda_{\rm L}^{(0)}\, {\rm d}t = {\rm d}t \int_M \rho(x) \Big\{ \tfrac{1}{2} \langle\, v(x), v(x)\, \rangle - \epsilon - c^2 \Big\}\, {\rm d}^3 x\,.$$

First two terms in the bracket { } are equivalent with those of (109). The third term $-c^2 {\rm d}t\, \rho {\rm d}^3 x = -c^2 {\rm d}t\, {\rm d}^3 \boldsymbol{a}$ is necessary for the Lorentz invariance in which t is transformed simultaneously with x. But it may be omitted for the Galilean invariance, since the mass element ${\rm d}^3 \boldsymbol{a}$ is invariant and $t' = t$ in LGT. Hence the third term may be neglected, and the first two terms are understood to be invariant with respect to the Galilean transformation. This invariance is valid locally at each point and time.

Remmaining three terms of (109) do not influence the outcome of the action principle, as explained in §2.5 and 3.4, since those are integrated with respect to the time. It is remarkable that those three have transformation invariance locally as well, *i.e.* LGT-invariance. In fact, we can write as $g\rho = \rho$, since the density ρ is a scalar field. Setting $u = \rho$, the transformation (118) is written as $\rho'(x') = g\rho(x) = \rho(x)$, showing its gauge inavriance. Other scalar fields such as $s(x)$, $\phi(x)$ or $\psi(x)$ are also gauge inavriant as well. Further more, LGT-invariance of the operator ${\rm D}_t$ is already shown by (115). Thus, it is found that the third and forth terms have the LGT-invariance.

In regard to the last term of (109), we note the following equality from (38) and (49):

$$\langle \mathcal{L}_W \boldsymbol{A},\, \boldsymbol{\omega}\, \rangle\, {\rm d}^3 \boldsymbol{x} = \langle \partial_\tau \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle\, {\rm d}^3 \boldsymbol{a}. \qquad (120)$$

The right hand side is independent of τ and a function of $\boldsymbol{a}$ only, which is shown in §6.5 by (105) and (107). Therefore, it is gauge invariant because $\boldsymbol{a}$ is invariant of LGT and $\partial_\tau = {\rm D}_t$ is invariant too. Therefore, the last term has the LGT-invariance.

Thus, it has been found that Λ of (109) has the translation symmetry.

Rotational symmetry of fluid motions is represented by the rotation group $SO(3)$. Infinitesimal rotations are expressed by Lie algebra $\mathbf{so}(3)$. The algebra $\mathbf{so}(3)$ is three-dimensional with a set of bases (e_1, e_2, e_3) satisfying the following commutation law:

$$[\, e_j,\, e_k\,] = \varepsilon_{jkl}\, e_l\,, \tag{121}$$

where ε_{jkl} is the third-order completely skew-symmetric tensor. The bases (e_1, e_2, e_3) are represented by the following skew-symmetric matrices:

$$e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{122}$$

A rotation operator is defined by $\theta = \theta_k e_k$:

$$\theta = (\theta_{ij}) \;\equiv\; \theta_k\, e_k = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix},$$

where θ_k $(k = 1, 2, 3)$ are infinitesimal parameters, and θ_{ij} is a skew-symmetric matrix with $\theta_{ij} = -\theta_{ji}$. Infinitesimal rotation of a displacement vector $\boldsymbol{s} = (s_1, s_2, s_3)$ is given by θs, which is also represented by the vector product $\hat{\theta} \times \boldsymbol{s}$, with an infinitesimal angle vector $\hat{\theta} = (\theta_1,\, \theta_2,\, \theta_3)$.

Consider local rotation around an arbitrarily-chosen reference point $\boldsymbol{x}_0$ within the fluid. Neighboring points are represented as $\boldsymbol{x}_0 + \boldsymbol{s}$ with $\boldsymbol{s}$ being an infinitesimal coordinate vector $\boldsymbol{s}$. Consider local rotation of the neighborhood of $\boldsymbol{x}_0$, expressed by $\boldsymbol{s} \to \boldsymbol{s}' = \boldsymbol{s} + \delta\boldsymbol{s}$. In a rotational transformation, $\delta\boldsymbol{s}$ is represented by

$$\delta\boldsymbol{s} = \theta\, \boldsymbol{s} = \theta_k e_k\, s, \tag{123}$$

equivalently expressed by $\delta\boldsymbol{s} = \hat{\theta} \times \boldsymbol{s}$, with $\hat{\theta} = \hat{\theta}(t)$ an infinitesimal rotation angle.

This is regarded as a gauge transformation from a point $\boldsymbol{s}$ of a frame F to a point $\boldsymbol{s}'$ of another frame F' at identical times:

$$\boldsymbol{s}'(\boldsymbol{s}, t) = \boldsymbol{s} + \delta\boldsymbol{s}(\boldsymbol{s}, t), \qquad t' = t. \tag{124}$$

From this, the velocity $\boldsymbol{v} = \mathrm{D}_t \boldsymbol{s}$ is transformed as follows:

$$\boldsymbol{v}'(\boldsymbol{s}') = (\mathrm{D}_t)'\boldsymbol{s}' = \mathrm{D}_t(\boldsymbol{s} + \delta\boldsymbol{s}) = \boldsymbol{v}(\boldsymbol{s}) + \delta\boldsymbol{v}(\boldsymbol{s}), \tag{125}$$

$$\delta\boldsymbol{v}(\boldsymbol{s}) = \mathrm{D}_t\delta\boldsymbol{s} \equiv \partial_t\delta\boldsymbol{s} + (\boldsymbol{v}\cdot\nabla_s\Big)\delta\boldsymbol{s} = (\partial_t\theta)\boldsymbol{s} + \theta\boldsymbol{v}, \tag{126}$$

where $\nabla_s = (\partial/\partial s_i)$. Like the LGT of §8.1, two position vectors $\boldsymbol{s}$ and $\boldsymbol{s}'$ denote an identical point expressed differently in two different frames F and F'. Their coordinates are different, but their particle coordinates shares the same $\boldsymbol{a}$. This means that the frame F' is rotated by an angle $-\hat{\theta}$ with respect to the frame F sharing the same origin $\boldsymbol{x}_0$.

From (125) and (126), at the origin $\boldsymbol{x}_0$ ($\boldsymbol{s} = 0$) we have the velocity $v = \boldsymbol{v}(0)$, and its variation

$$\delta\boldsymbol{v}(0) = \theta\, v = \theta_k e_k v. \tag{127}$$

Transformations of $\boldsymbol{s}$ and $\boldsymbol{v}$ are given from (123) and (127) by

$$s' = r\, s\,, \qquad s'_i = r_{ij}\, s_j, \tag{128}$$

$$v' = r\, v, \qquad v'_i = r_{ij}\, v_j, \tag{129}$$

where $r = I + \theta$ with $I = \delta_{ij}$ (unit matrix) and $\theta = \theta_{ij}$, *i.e.* $r_{ij} = \delta_{ij} + \theta_{ij}$. It is found that the pair set of s and v have a common transformation property like that of x and u of 8.3 ($\boldsymbol{a}$).

8.6 Rotation symmetry

Inner product is invariant by rotational transformations of $SO(3)$ (by definition). Choosing an element $r = (r_{ij}) \in \mathcal{G} = SO(3)$ and taking two vectors u and v, we write the transformation by $u' = ru = r_{ij}u_j$ and $v' = rv = r_{ij}v_j$ respectively. Their inner product $\langle u,\, v\rangle = \delta_{ij}u_i v_j$ is invariant with respect to the rotational transformation (δ_{ij}: Kronecker's delta). In fact, we have

$$\langle u',\, v'\rangle = \langle ru,\, rv\rangle = \langle u,\, v\rangle,$$

which reduces to the relation of matrix elements (of $SO(3)$): $\delta_{ij}r_{ik}r_{jl} = \delta_{kl}$. This is valid within linear approximation to $r_{ij} = \delta_{ij} + \theta_{ij}$ of the previous section. Owing to the antisymmetry of θ_{ij} and their smallness, we have

$$\delta_{ij}r_{ik}r_{jl} = \delta_{kl} + \theta_{kl} + \theta_{lk} + O(|\theta|^2) = \delta_{kl} + O(|\theta|^2).$$

This leads immediately to the rotational invariance of the first term $\langle \boldsymbol{v},\, \boldsymbol{v}\rangle$ of (109). In addition, its last term represented by (120) is also rotationally invariant, because it is of the form of an inner product. The invariance is also obvious since it is a function of $\boldsymbol{a}$ only and $\boldsymbol{a}$ is unchanged by rotational gauge transformations.

Scalar fields such as ρ, s, ϵ, ϕ and ψ are invariant. The reason is as follows. In a rotational transformation r, a scalar field $u(x)$ is transformed to $u' = ru = u$ (no change of its functional form), and the equation (118) means the invariance $u'(x') = u(x)$.

Remaining issue is about the operator D_t, defined by $\mathrm{D}_t = \partial_t + v_k \, \partial_k$. The second term is a scalar product, hence it is rotationally invariant (omitting its mathematical detail of verification). Therefore, we have the invariance: $\mathrm{D}'_{t'} = \mathrm{D}_t$.

Summarizing the aboves, it is found that Λ of (109) has the rotation symmetry.

9 Summary and discussions

Vaiational principle is reformulated, according to the gauge theory of theoretical physics, for the motions of an ideal fluid on the basis of newly defined Lagrangians and covariant derivative. This new variational formulation is superior to most traditional ones in the following three aspects. (i) The formulation is based on the symmetries (of translation and rotation) of the flow fields. (ii) It does not include Lagrange mutipliers to impose constraints. This is an advatage because physical meaning of the mutipliers are not clear. And (iii) the rotational part of velocity field is taken into account naturally by a new Lagrangian. Thus, the formulation is reasonable in physical sense and consistent as a whole for description of flow fields of an ideal fluid.

The Lagrangian functional is defined initially by a combination of total kinetic energy and internal energy in the space of particle coordinates (denoted as $\boldsymbol{a}$ space), and the action is given by its time integral. Noether's theorem leads to the equation of motion and an energy equation. In most traditional formulations, the continuity equation and an entropy equation are added as constraints by using Lagrange multipliers to the action integral. However, in the new formulation, those equations are derived from the variational principle, rather than being given as constraints. To that end, additional Lagrangians are introduced by symmetry consideration.

In the present formulation, the newly added Lagrangians are determined such that it is invariant with respect to both translational and rotational transformations, where the following three properties are taken into consideration: (a) mass is an invariant of motion, (b) entropy per unit mass is another invariant of motion owing to the definition of an ideal fluid, and (*c*) vorticity $\boldsymbol{\Omega}_a$ in the $\boldsymbol{a}$-space is invariant, which is derived by requirement of gauge invariance of the action in the $\boldsymbol{a}$-space. The new three Lagrangians thus introduced are of such forms that the action integrals associated with those Lagrangians are integrated with respect to time in the $\boldsymbol{a}$-space. So that, the newly added terms do not influence the Euler-Lagrange equation in the $\boldsymbol{a}$-space. However, from the action principle in the space of Eulerian coordinates $\boldsymbol{x}$, the continuity equation and an entropy equation are derived, since the time derivative in the Lagrange space is replaced with the covariant derivative (*i.e.* the material derivative) including velocity components, and the mass density is a function of $\boldsymbol{x}$ and time t.

The third property (*c*) is important in the sense that the vorticity is a gauge field associated with the rotation symmetry of the flow field, and that the vorticity equation is derived from the Lagrangian L_A associated with (*c*). Thus it is seen that the vorticity equation is the equation for the gauge field of rotation symmetry.

The fact that vorticity field must be considered independently of the velocity field is necessary from the following property. In each space-time of the Lagrangian and Eulerian descriptions, three vectors of velocity, acceleration and vorticity can be determined. Thus, we have nine tranformation relations at each space-time between each pair of the three vectors, from which nine coefficients $\partial x^k/\partial a^l$ of transformation can be fixed locally (at each space-time point). With using the local nine coefficients, transformation between $\boldsymbol{a}$-space and $\boldsymbol{x}$-space is determined locally and uniquely.

It was shown in §6 that the Lagrangian L_A and the helicity $\mathcal{H}$ are related to each other in the sense that, when one exists, then the other also exists. It is verified that both of the integrals are time-invariant, by using the time-invariance of $\boldsymbol{\Omega}_a$. In traditional fluid mechanics, although the helicity $\mathcal{H}$ was investigated in detail, the Lagrangian L_A has never been considered explicitly.

Thus, it has been shown that the present formulation is consistent as a whole to describe flow fields of an ideal fluid.

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